THE CHINESE UNIVERSITY OF HONG KONG
MATH4010 Suggested solutions to homework 1

If you find any mistakes or typos, please report them to ypyang@math.cuhk.edu.hk

4.7. Solution. T is linear: $\forall \alpha, \beta \in \mathbb{R}, \forall x_1(t), x_2(t) \in C[0,1], \forall t \in [0,1]$,

$$\begin{aligned} T(\alpha x_1 + \beta x_2)(t) &= \int_0^1 k(t,u)(\alpha x_1(u) + \beta x_2(u))\, du \\ &= \alpha \int_0^1 k(t,u) x_1(u)\, du + \beta \int_0^1 k(t,u) x_2(u)\, du \\ &= \alpha T(x_1)(t) + \beta T(x_2)(t). \end{aligned}$$

Therefore, $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$.

T maps $C[0,1]$ to $C[0,1]$: For any $x(t) \in C[0,1]$, there exists $M > 0$ such that $|x(t)| \leq M, t \in [0,1]$. $\forall t_0 \in [0,1], \forall \varepsilon > 0$, since $k(t,u)$ is continuous on $[0,1]^2$, there exists $\delta > 0$ such that if $|t - t_0| < \delta$ and $t \in [0,1]$ then $|k(t,u) - k(t_0,u)| < \dfrac{\varepsilon}{M}$. Then whenever $|t - t_0| < \delta$ and $t \in [0,1]$ we have

$$\begin{aligned} |Tx(t) - Tx(t_0)| &= \left| \int_0^1 k(t,u) x(u)\, du - \int_0^1 k(t_0,u) x(u)\, du \right| \\ &\leq \int_0^1 |k(t,u) - k(t_0,u)| \cdot |x(u)|\, du \\ &< \int_0^1 \frac{\varepsilon}{M} \cdot M\, du = \varepsilon. \end{aligned}$$

Therefore, Tx is a continuous function on $[0,1]$.

4.9. Solution. $\forall x \in l^p$,

$$\|T_l x\|_p = \left(\sum_{k=2}^{\infty} |x_k|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} = \|x\|_p.$$

So we have that T_l is bounded and $\|T_l\| \leq 1$. Take $x = e_2 = (0,1,0,0,\cdots)$ and we can get

$$\|T_l x\|_p = \|(1,0,0,\cdots)\|_p = 1 \leq \|T_l\| \|x\|_p = \|T_l\|.$$

Therefore, $\|T_l\| = 1$.

Similarly,

$$\|T_r x\|_p = \left(0^p + \sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} = \|x\|_p.$$

It follows that T_r is bounded with $\|T_r\| = 1$.

4.10. Solution. Let $y \in \widetilde{X}$, there exists a sequence $\{x_n\} \subset X$ such that $x_n \to y$. Then we have $\|f(x_m) - f(x_n)\| = \|f(x_m - x_n)\| \leq \|f\| \|x_m - x_n\|$. Since convergent sequences are Cauchy, $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. This implies that $\{f(x_n)\}$ is Cauchy and $\lim\limits_{n\to\infty} f(x_n)$ exists.

Define $\widetilde{f}(y) = \lim_{n\to\infty} f(x_n)$. We claim that $\widetilde{f}$ is well defined. Suppose $\{y_n\}$ is an arbitrary sequence such that $y_n \to y$. Then $\|f(y_n) - f(x_n)\| \leq \|f\|\|y_n - x_n\| \to 0$ as $n \to \infty$. This implies

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(x_n) = \widetilde{f}(y).$$

$\widetilde{f}$ is linear: $\forall x, y \in \widetilde{X}, \forall \alpha, \beta \in \mathbb{R}$, there exist sequences $\{x_n\}, \{y_n\} \subset X$ such that $x_n \to x$, $y_n \to y$ as $n \to \infty$. Clearly $\alpha x_n + \beta y_n \to \alpha x + \beta y$. Therefore,

$$\widetilde{f}(\alpha x + \beta y) = \lim_{n\to\infty} f(\alpha x_n + \beta y_n) = \alpha \lim_{n\to\infty} f(x_n) + \beta \lim_{n\to\infty} f(y_n) = \alpha \widetilde{f}(x) + \beta \widetilde{f}(y).$$

$\widetilde{f}|_X = f$: $\forall x \in X$, choose $\{x_n\} = (x, x, \cdots)$ and then $x_n \to x$. So that

$$\widetilde{f}(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x) = f(x).$$

$\widetilde{f}$ is bounded: for any $y \in \widetilde{X}$, we have

$$\left|\widetilde{f}(y)\right| \leq \lim_{n\to\infty} |f(x_n)| \leq \lim_{n\to\infty} \|f\|\|x_n\| = \|f\| \lim_{n\to\infty} \|x_n\| = \|f\|\|y\|.$$

$\widetilde{f}$ is unique: Suppose there is another extension f_1. Then

$$\widetilde{f}(y) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f_1(x_n) = f_1(y).$$

Finally, let $n \to \infty$ in $\|f(x_n)\| \leq \|f\|\|x_n\|$ and we get $\|\widetilde{f}(y)\| \leq \|f\|\|y\|$. Hence $\|\widetilde{f}\| \leq \|f\|$. It's clear that $\|\widetilde{f}\| \geq \|f\|$ because the norm, defined as a supremum, must be non-decreasing in an extension. Together we have $\|\widetilde{f}\| = \|f\|$.